

A remark on Dickey's stabilizing chain

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Abstract

We observe that Dickey's stabilizing chain can be naturally included into two-dimensional chain of infinitely many copies of equations of KP hierarchy.

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1. Introduction

In his article [1] Dickey introduced the so-called stabilizing chain of truncated Kadomtsev-Petviashvili (KP) hierarchies. The latter can be formulated in the language of formal pseudo-differential dressing operators (dressing Ψ DO's) $W_i = 1 + \sum_{m=1}^i w_{im} \partial^{-m}$ which are forced to be connected by relations

$$(\partial + u_i)W_i = (\partial + v_{i+1})W_{i+1} \quad (1)$$

for $i \geq 0$. One requires that solution of (1) such that w_{ii} is not identically equal to zero. Evolution equations of KP hierarchy are given by

$$\partial_s W_i = (Q_i^s)_+ W_i - W_i \partial^s, \quad (2)$$

for $s \geq 2$, where $Q_i \equiv W_i \partial W_i^{-1}$ and $\partial_s \equiv \partial / \partial t_s$. Remember that the subscript $+$ means taking only nonnegative powers of ∂ in pseudo-differential operator under consideration. To complete the description of stabilizing chain we need to add evolution equations for "gluing" fields u_i and v_i :

$$\partial_s u_i = -\text{res}_\partial \left((\partial + u_i)(Q_i^s)_+ (\partial + u_i)^{-1} \right), \quad (3)$$

$$\partial_s v_i = -\text{res}_\partial \left((\partial + v_i)(Q_i^s)_+ (\partial + v_i)^{-1} \right). \quad (4)$$

Remember that $\text{res}_\partial (\sum a_m \partial^m) \equiv a_{-1}$. It was shown in [1] that equations (1-4) are well defined. Moreover, general solution of stabilizing chain is shown to be given in terms of Wronskians

$$W_i = \frac{1}{\tau_{i1}} \begin{vmatrix} y_{0i} & \cdots & y_{i-1,i} & 1 \\ y'_{0i} & \cdots & y'_{i-1,i} & \partial \\ \vdots & \vdots & \vdots & \vdots \\ y_{0i}^{(i)} & \cdots & y_{i-1,i}^{(i)} & \partial^i \end{vmatrix} \partial^{-i},$$

$$u_i = -\partial \ln \frac{\tau_{i+1,2}}{\tau_{i1}}, \quad v_i = -\partial \ln \frac{\tau_{i2}}{\tau_{i1}},$$

where $\tau_{i1} = \text{Wr}[y_{0i}, \dots, y_{i-1,i}]$ and $\tau_{i2} = \text{Wr}[y_{0,i-1}, \dots, y_{i-1,i-1}]$. By definition, the set $\{y_{0i}, \dots, y_{i-1,i}\}$ is the basis of the kernel for differential operator $P_i \equiv W_i \partial^i$. In what follows, we set $y_{i-1,i-1} \equiv y'_{i-1,i}$. Functions y_{kl} are forced to be solutions of hierarchy evolution equations $\partial_s y = \partial^s y$. As is known any analytic solution of this hierarchy can be presented as series over Schur polynomials

$$y = \sum_{m \geq 0} c_m p_m(x, t_2, t_3, \dots)$$

Let us remember that Schur polynomials are defined through the relation

$$\exp \left(\sum_{s \geq 0} t_s z^s \right) = \sum_{m \geq 0} p_m z^m, \quad \text{where } t_1 = x$$

and have, in virtue of their definition, following easily verified properties:

$$p_m(x, 0, 0, \dots) = x^m / m! \quad \text{and} \quad \partial_s p_m = \partial^s p_m = p_{m-s}.$$

As was shown in [1] the sequence $\{\tau_{i1}\}$ has the property of stabilization with respect to gradation which is defined by the rule: $[t_k] = k$. Namely, if one choose

$$y_{ki} = (-1)^k \left(p_{i-k-1} + c_1^{(k)} p_{i-k} + c_2^{(k)} p_{i-k+1} + \cdots \right),$$

for $k = 0, \dots, i-1$, then any term of weight l do not depend on i when $i \geq l$. In this case, one says that the sequence $\{\tau_{i1}\}$ has the stable limit. Moreover, with special choice of constants $c_m^{(k)}$ this stable limit yields expression for Kontsevich integral [2].

In the next two sections we present our observation that it is quite natural to put equations (1-4) into two-dimensional chain of KP hierarchies.

2. Two-dimensional chain of KP hierarchies

2.1. Two-dimensional chain of dressing Ψ DO's

Here we construct two-dimensional chain of truncated dressing Ψ DO's $\{W_{ij}\}$ related with each other by some suitable relations.

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With infinite set of suitable constants $\{c_{kl} : k, l \in \mathbb{Z}, k \geq 0\}$ we define collection of analytic functions

$$\bar{y}_{kl} = \sum_{m \geq 0} c_{k,l-m} \frac{x^m}{m!}. \quad (5)$$

Obviously, by definition, $\bar{y}'_{kl} = \bar{y}_{k,l-1}$. Let us define

$$\tau_{ij} \equiv \text{Wr}[\bar{y}_{0,i+1-j}, \dots, \bar{y}_{i-1,i+1-j}]$$

and an infinite set of differential operators

$$P_{ij} = \partial^i + \sum_{m=1}^i w_{im}^j \partial^{i-m} = \frac{1}{\tau_{ij}} \begin{vmatrix} \bar{y}_{0,i+1-j} & \cdots & \bar{y}_{i-1,i+1-j} & 1 \\ \bar{y}'_{0,i+1-j} & \cdots & \bar{y}'_{i-1,i+1-j} & \partial \\ \vdots & \vdots & \vdots & \vdots \\ \bar{y}_{0,i+1-j}^{(i)} & \cdots & \bar{y}_{i-1,i+1-j}^{(i)} & \partial^i \end{vmatrix}$$

for $i, j \in \mathbb{Z}, i \geq 0$. Require that w_{ii}^j is not identically equal to zero for any values of i and j . This is equivalent to the fact that P_{ij} has not $y = \text{const}$ as a solution. In what follows, it will be useful following technical proposition.

Lemma 1. *In virtue of their definition, operators P_{ij} satisfy equations*

$$(\partial + v_{i+1,j})P_{i+1,j} = P_{i+1,j+1}\partial, \quad (\partial + u_{ij})P_{ij} = P_{i+1,j+1} \quad (6)$$

with

$$v_{ij} = -\partial \ln \left(\frac{\tau_{i,j+1}}{\tau_{i,j}} \right), \quad u_{ij} = -\partial \ln \left(\frac{\tau_{i+1,j+1}}{\tau_{ij}} \right). \quad (7)$$

PROOF. The first relation in (6) follows from the fact that $\ker P_{i+1,j+1}$ is defined as a linear spanning of derivatives of the functions which belong to $\ker P_{i+1,j}$. Functions $\bar{y}'_{0,i+2-j}, \dots, \bar{y}'_{i,i+2-j}$ are linearly independent. Otherwise, $y = \text{const}$ will belong to $\ker P_{i+1,j}$. Moreover, we have the relation

$$(\partial + v_{i+1,j})P_{i+1,j}(1) = 0$$

hold. From

$$P_{i+1,j}(1) = \frac{\text{Wr}[\bar{y}_{0,i+2-j}, \dots, \bar{y}_{i,i+2-j}, 1]}{\tau_{i+1,j}} = (-1)^{i+1} \frac{\tau_{i+1,j+1}}{\tau_{i+1,j}}$$

we derive expression for v_{ij} in (7). The second relation in (6) follows from the fact that all basic functions of $\ker P_{i+1,j+1}$ except for $\bar{y}_{i,i+1-j}$ belong to $\ker P_{ij}$. In addition, we have

$$(\partial + u_{ij})P_{ij}(\bar{y}_{i,i+1-j}) = (\partial + u_{ij}) \left(\frac{\tau_{i+1,j+1}}{\tau_{ij}} \right) = 0.$$

The latter gives corresponding expression for u_{ij} in (7). Therefore lemma is proved.

As a consequence of (6), we have two equations

$$(\partial + v_{i+1,j})W_{i+1,j} = W_{i+1,j+1}\partial, \quad (\partial + u_{ij})W_{ij} = W_{i+1,j+1}\partial \quad (8)$$

for $\Psi\text{DO's}$ $W_{ij} \equiv P_{ij}\partial^{-i}$. So, we can think of u_{ij} and v_{ij} as “gluing” variables which relate $\Psi\text{DO's}$ of special truncated form on two-dimensional chain. Two formulas in (8) define shifts $(i, j) \rightarrow (i, j+1)$ and $(i, j) \rightarrow (i+1, j+1)$, respectively. As a consequence, of these relations we see that W_{ij} also satisfies the relation

$$(\partial + u_{ij})W_{ij} = (\partial + v_{i+1,j})W_{i+1,j}. \quad (9)$$

which manages the shift $(i, j) \rightarrow (i+1, j)$. We see that this equation is nothing else but (1). The only difference is that dressing operators W_i in (9) are parameterized by additional discrete variable j .

2.2. Two-dimensional chain of KP hierarchies

We know that if one replaces the basis $\{x^m/m!\}$ by that of Schur polynomials $\{p_m(x, t_2, t_3, \dots)\}$ in (5), that is,

$$\bar{y}_{kl} \rightarrow y_{kl} = \sum_{m \geq 0} c_{k,l-m} p_m,$$

then each W_{ij} automatically will be solution of KP hierarchy (2) (see, for example [3]), while the sequence of dressing operators along shifts $(i, j) \rightarrow (i+1, j+1)$ and $(i, j) \rightarrow (i, j+1)$, due to (8) is nothing else but the semi-infinite 1-Toda lattice (the discrete KP hierarchy) with initial condition $W_{0j} = 1$. Then “gluing” variables u_{ij} and v_{ij} , by their construction, automatically satisfy equations [1]

$$\partial_s u_{ij} = -\text{res}_\partial \left((\partial + u_{ij})(Q_{ij}^s)_+ (\partial + u_{ij})^{-1} \right),$$

$$\partial_s v_{ij} = -\text{res}_\partial \left((\partial + v_{ij})(Q_{ij}^s)_+ (\partial + v_{ij})^{-1} \right).$$

3. Conclusion

In this brief note we have shown how Dickey’ stabilizing chain (1-4) can be included into two-dimensional lattice of KP hierarchies. One learns from this presentation, that, the latter in a sense can be viewed as a superposition of two compatible discrete KP hierarchies.

References

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